Backward Stochastic Differential Equation and Exact Controllability of Stochastic Control Systems*

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Introduction

This paper is concerned with the structures of the backward stochastic differential equations (BSDE)\(^{1-9}\). We provide an approach to the determination of whether a formulation of certain types of BSDE is well-posed in the existence viewpoint. As an application of our results, we give a necessary condition of terminal controllability for a stochastic control system. To linear stochastic control systems, we introduce a pure algebraic criterion of necessary and sufficient condition for a system to be exactly controllable.

The BSDE discussed in this paper has the following form:

\[
\begin{align*}
- dy_t &= f(y_t, z_t, t) \, dt - g(y_t, z_t, t) \, dW_t, \quad 0 \leq t \leq T, \\
y_T &= \xi,
\end{align*}
\]  

(1)

where \(\{W_t, t \geq 0\}\) is a standard Wiener process (Brownian motion); for each \((y, z)\), \(f(y, z, \cdot)\) and \(g(y, z, \cdot)\) are given \(\mathcal{F}_t\)-adapted processes. The formulation of our problem is, for any given terminal condition \(y_T = \xi\) which is \(\mathcal{F}_T\)-measurable, to find a pair of \(\mathcal{F}_t\)-adapted processes \((y(\cdot), z(\cdot))\) that solves (1). We will look for conditions under which solutions of (1) exist. This problem is called E-well-posedness, where E stands for "existence".

The E-well-posedness of BSDEs is related closely to the following "exact controllability" problem: for any given initial condition \(x \in \mathcal{H}\) and a \(\mathcal{F}_T\)-measurable terminal condition \(\xi\), we look for an admissible control \(u(\cdot)\) such that

\[
dx_t = b(x_t, u_t) \, dt + \sigma(x_t, u_t) \, dW_t, \quad x_0 = x, \quad x_T = \xi, \quad 0 \leq t \leq T.
\]  

(2)

We will show that our results of E-well-posedness can be applied to this controllability problem. When the above control system is linear, we can obtain an algebraic criterion of the necessary and sufficient conditions.

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To our knowledge, the results of exact controllability obtained in this paper are novel. Recent study of controllability (in weaker senses) of stochastic control systems may be found in Refs. [10] and [11]. The controllability in the weakest sense, called influentiability, will be discussed in our forthcoming paper.

According to the mathematical model of the so-called “European option” (see, for example, Refs. [12] and [13]) and stochastic differential utility in mathematical finance (see Refs. [14] and [15]), \( y \) in (1) (or \( x \) in (2)) may be interpreted as an evaluation process of the “fair price”, whereas \( z \) in (1) (or \( v \) in (2)) may be interpreted as the related consumption and portfolio processes. We treat this problem with BSDE approach. One of the advantages of this approach is that it can be easily extended to the multi-dimensional and/or nonlinear situation.

The paper is organized as follows. The main notations and preliminaries on BSDE are given in Sec. 1. Sec. 2 is devoted to the E-well-posedness for a general formulation of BSDE in which Lemma 2.2, the “non-representation theorem”, plays the fundamental role. In Sec. 3, we discuss the exact controllability for general control systems.

1 Notations and Preliminaries

Let \((\Omega, \mathcal{F}, P)\) be a probability space endowed with a filtration \(\mathcal{F}_t, t \geq 0\). Let \((W_t, t \geq 0)\) be a \(d\)-dimensional standard Wiener process (Brownian motion) in this space, \(W = (W^1, \ldots, W^d)\).

We assume that the filtration \(\mathcal{F}_t\) is generated by this Wiener processes \(W\)

\[ \mathcal{F}_t = \sigma\{W_s, s \leq t\}. \]

All processes mentioned in this paper are assumed to be \(\mathcal{F}_t\)-adapted. The set of all \(\mathbb{R}^n\)-valued square integrable processes, i.e.

\[ E \int_0^T |v_t|^2 \, dt < \infty, \]

is denoted by \(L^2_{\mathcal{F}}(0, T; \mathbb{R}^n)\).

In Ref. [1], we introduced a BSDE of the following type:

\[
\begin{aligned}
-dy_t &= g_0(y_t, k_t, t) \, dt - \sum_{i=1}^d (g_i(y_t, k_t) - k^i_t) \, dW^i_t, \quad 0 \leq t \leq T, \\
}\end{aligned}
\]

\[ y_T = \xi. \]

The formulation of the problem is to find a unique pair of processes

\((y, k) = (y^1, \ldots, k^d)\)
satisfying (3).

Let \( g = (g_1, \cdots, g_d) \). If we set

\[
 f(y, z, t) = g_0(y, z - g(y, t), t).
\]

Then the above formulation is equivalent to finding a pair of processes \((y, z) = (y_1, z^1, \cdots, z^d)\) which solves

\[
\begin{cases}
  -dy_i = f(y_i, z_i, t) dt - \sum_{i=1}^d z_i^i dW^i, & 0 \leq t \leq T, \\
  y_T = \xi.
\end{cases}
\]

(4)

In fact, we have a relation \( z_i = k_i + g(y_i, t) \). For this reason, we usually consider the relatively simple form (4). We have the following existence and uniqueness result (see Ref. [1] or Ref. [5] for proofs).

**Theorem 1.1.** For each \((y, z) \in \mathbb{R}^m \times \mathbb{R}^{m \times d}\) let \( f(y, z, \cdot) \) be a process in \( L^2_{\mathcal{F}}(0, T; \mathbb{R}^m) \), and let \( f(y, z, t) \) be Lipschitz with respect to \((y, z)\) uniformly in \( t \in [0, T] \). Then for any given terminal condition \( y_T = \xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^m) \), there exists a unique pair of processes

\[
(y, z) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m \times d}),
\]

which solves the BSDE (4).

**Remark 1.1.** Since formulations (3) and (4) are equivalent, under assumptions that \( g_i, i = 0, 1, \cdots, d \), are Lipschitz in \( y \) and \( k \), there exists a unique pair of processes \((y(\cdot), k(\cdot))\) satisfying (3).

## 2 Structure Consideration of BSDE

For notational brevity, we only consider one-dimensional Brownian motion: \( d = 1 \). The case for multi-dimensional Brownian motion can be treated analogously. The BSDE (4) thus becomes

\[
\begin{cases}
  -dy_i = f(y_i, z_i, t) dt - z_i dW, & 0 \leq t \leq T, \\
  y_T = \xi.
\end{cases}
\]

(5)

Here \( z \) is an \( \mathbb{R}^m \)-valued unknown process.

We now briefly explain, under the assumptions of Theorem 1.1, how to prove the uniqueness of the BSDE (5). Let \((y^1, z^1)\) and \((y^2, z^2)\) be two solutions of (5) and let \((\tilde{y}, \tilde{z})\) be their difference. Using Itô's formula for \(|\tilde{y}|^2\) yields

\[
E|\tilde{y}|^2 + E \int_t^T |\tilde{z}|^2 ds \leq 2CE \int_t^T |\tilde{y}| \cdot (|\tilde{y}| + |\tilde{z}|) ds.
\]

Thus, from a simple algebraic inequality \( 2C \leq 2C^2 a^2 + \frac{1}{2} b^2 \) it follows that
\[
E|\tilde{y}|^2 + \frac{1}{2} \int_0^T |\tilde{z}|^2 ds \leq 2C_k E \int_0^T |\tilde{y}|^2 ds.
\]

Now ignoring the \( \tilde{z} \)-term and applying Gronwall's inequality we have \( \tilde{y} = 0 \). Then \( \tilde{q} = 0 \).

From the above we can see that (5) possesses a natural structure in the viewpoint of uniqueness. In this section we will show that this kind of structure is also "almost necessary" in the viewpoint of existence.

Consider the following general form of BSDE

\[
\begin{align*}
-dy_t &= f(y_t, z_t, t) \, dt - g(y_t, z_t, t) \, dW_t, \\
y_T &= \xi.
\end{align*}
\tag{6}
\]

Here for each \((y, z) \in \mathcal{A}^m \times \mathcal{A}^k\), \(f(y, z, \cdot)\) and \(g(y, z, \cdot)\) are processes in \(L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)\).

We note that in this formulation \( z \) is \( \mathcal{A}^k \)-valued. Here \( k \) may be different from \( m \) which is the dimension of \( y \).

We introduce the following notation.

**Definition 2.1** BSDE (6) is called E-well-posed if for any given \( \xi \in L^2(\Omega, \mathcal{F}_T, P, \mathbb{R}^m) \) there exists at least one pair of processes

\[(y, z) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m) \times L^2(0, T; \mathbb{R}^k)\]

that solve (6).

Here we will give some necessary conditions for E-well-posed systems. As a co-product of this discussion, we will give an algebraic criterion of exact controllability for linear stochastic control systems (see the next section).

In Ref. [1], we used a well-known square-integrable martingale representation theorem in terms of Brownian motion to prove the existence of BSDE (5). This theorem tells us that for any square-integrable \( \mathcal{F}_T \)-measurable random variable \( \xi \in L^2(\Omega, \mathcal{F}_T, P, \mathbb{R}) \), one can find a unique process \( q \in L^2(0, T; \mathbb{R}) \), such that

\[
\xi = E\xi + \int_0^T q_s dW_s.
\]

To get a deep-going understanding of the E-well-posedness of a BSDE, we need to introduce a simple but useful lemma, which gives a "non-representation" result.

We set

\[
\phi_i = \begin{cases} 
+1, & \text{when } t \in [(1-2^{-i}) T, (1-2^{-(i+1)}) T), \\
-1, & \text{otherwise}
\end{cases}, \quad i = 0, 1, \ldots
\]

(7)

It is seen that there exists a constant \( \beta > 0 \) such that
\[
\int_t^T |\phi_s-c|^2 ds \geq 4\beta (T-t), \quad \forall (c, t) \in \mathcal{A} \times [0, T).
\] (8)

Setting
\[
\zeta = \int_0^T \phi_s dW_s,
\] (9)
we have

**Lemma 2.1.** It is impossible to find \((a, b) \in L^2_\mathcal{F} (0, T; \mathcal{A}) \times L^2(0, T; \mathcal{A})\) and \(x \in \mathcal{A}\) with
\[
\lim_{t \to T} E|b_t-b_T|^2 = 0,
\] (10)
such that
\[
\zeta = x + \int_0^T a_t ds + \int_0^T b_t dW_t.
\] (11)

**Proof.** Suppose that on the contrary, (11) holds. It is easy to deduce that
\[
\int_t^T (\phi_s-b_s) dW_s = \int_t^T a_s ds - E^x \int_t^T a_s ds.
\]
Taking square and expectation on both sides of the above relation yields
\[
E \int_t^T |\phi_s-b_s|^2 ds = E \left( \int_t^T a_s ds - E^x \int_t^T a_s ds \right)^2 \leq E \left( \int_t^T a_s ds \right)^2 \leq (T-t) E \int_t^T |a_s|^2 ds.
\] (12)

But from (10) we know that there exists a positive constant \(\delta\), such that
\[
E|b_t-b_T|^2 \leq \beta, \quad \forall t \in [T-\delta, T)
\]
from which and Eq. (8), it follows that
\[
E \int_t^T |\phi_s-b_s|^2 ds \geq \frac{1}{2} E \int_t^T |\phi_s-b_s|^2 ds - E \int_t^T |b_T-b_s|^2 ds \geq 2\beta (T-t) - \beta (T-t),
\]
or
\[
E \int_t^T |\phi_s-b_s|^2 ds \geq \beta (T-t), \quad \forall t \in [T-\delta, T).
\]

This is contradictory to (12). The proof is completed.

We now proceed to consider a necessary condition of \(E\)-well-posedness of a BSDE
\[
\begin{cases}
-dy_t = f(y_t, z_t, t) dt - g(y_t, z_t, t) dW_t, & 0 \leq t \leq 0, \\
y_T = \xi.
\end{cases}
\] (13)
Here, we assume

**Hypothesis 2.1.** For each \((y, z) \in \mathbb{R}^m \times \mathbb{R}^k\), processes \(f(y, z, \cdot)\) and \(g(y, z, \cdot)\) are in \(L^2_T(0, T; \mathbb{R}^m)\), and

\[
\lim_{t \to T} E|g(y, z, t) - g(y, z, T)|^2 = 0.
\]

\(f(y, z, t)\) and \(g(y, z, t)\) satisfy the linear growth conditions with respect to \((y, z)\) uniformly in \(t \in [0, T]\). \(g\) is Lipschitzian with respect to \(y\) uniformly in \((z, t)\).

From Lemma 2.1, we have immediately

**Proposition 2.1.** A necessary condition for BSDE (13) to be E-well-posed is

\[
\begin{cases}
\text{for any } a \in \mathbb{R}^k \text{ and } b \in \mathbb{R}^m \text{ with } |b| = 1, \text{ there exists at least one pair } \\
(y, z) \in \mathbb{R}^m \times \mathbb{R}^k, \text{ such that the scalar product } b \cdot (g(y, z, t) - g(y, a, t)) \neq 0.
\end{cases}
\]  

(14)

**Proof.** We observe that any solution \((y, z)\) of (13) satisfies

\[
\lim_{t \to T} E|g(y, a, t) - g(y_T, a, T)|^2 = 0, \quad \text{and } f(y(\cdot), z(\cdot), \cdot) \in L^2_T(0, T; \mathbb{R}^m),
\]

(15)

where \(q\) is an arbitrary vector \(q \in \mathbb{R}^k\). If condition (14) is false, then one can find \((b, a) \in \mathbb{R}^m \times \mathbb{R}^k\) with \(|b| = 1\), such that

\[
b \cdot (g(y, z, t) - g(y, a, t)) = 0, \quad \forall (y, z) \in \mathbb{R}^m \times \mathbb{R}^k.
\]

We now set \(y_T = \zeta b\), where \(\zeta\) is defined in (9). If \((y(\cdot), z(\cdot))\) solves (9), then relation \(b \cdot y_T = \zeta\) gives

\[
\zeta = b \cdot y_0 - \int_0^T b \cdot f(y, z, \cdot) \, dt + \int_0^T b \cdot g(y, a, t) \, dW_t.
\]

But from Lemma 2.1 and (15), this is impossible. The proof is completed.

**Remark.** The necessary condition (14) implies automatically that the dimension of \(z\) must be larger than or equal to that of \(y\); \(k \geq m\).

In the case where \(g\) is linear in \(y\),

\[
f(y, z, t) = G_1 z + g_1(y, t),
\]

where \(G_1\) is an \(m \times k\)-matrix. Then (14) becomes a necessary as well as sufficient condition. Indeed in this case we have

**Corollary 2.1.** Let \(f\) and \(g_1\) be uniformly Lipschitzian with respect to \((y, z)\). Then (13) is E-well-posed if and only if \(\text{Rank} G_1 = m\).

**Proof.** The "only if" part follows directly from Proposition 2.1. We need only to prove the "if" part. Since \(\text{Rank} G_1 = m\), then there exists a \(k \times k\) invertible matrix \(M\), such that \(G_1 M = [I_m, 0]\). We set
$M^{-1}z = \begin{pmatrix} \mu \\ v \end{pmatrix}, \quad \mu \in \mathbb{R}^n, \quad v \in \mathbb{R}^{n-m}.$

Under this invertible transformation, Eq. (13) may be equivalently rewritten as

$$\begin{cases} -d y_t = h(y_t, \mu, v_t, t) \, dt - [g_t(y_t) + \mu_t] \, dW_t, & 0 \leq t \leq T, \\ y_T = \xi. \end{cases} \quad (16)$$

where

$$h(y, \mu, v, t) = f \left( y, M \begin{pmatrix} \mu \\ v \end{pmatrix}, t \right).$$

It is seen that $h$ is Lipschitzian in $y$, $v$ and $\mu$. But in Sec. 1 (see Remark 1.1), we know that there exists a unique pair $(y, \mu) \in L^2_T(0, T, \mathbb{R}^m)$ satisfying

$$\begin{cases} -d y_t = h(y_t, \mu_t, 0, t) \, dt - [g_t(y_t) + \mu_t] \, dW_t, & 0 \leq t \leq T, \\ y_T = \xi. \end{cases} \quad (17)$$

Thus $(y(\cdot), \mu(\cdot), 0)$ solves (17). From it follows immediately that

$$(y_t, \cdot) = \left( y_t, M \begin{pmatrix} \mu_t \\ 0 \end{pmatrix} \right), \quad t \in [0, T],$$

is a solution of (13) with $g(y, z, t) = G, t + g_t(y, t)$. The proof is completed.

Remark. The assumption of the above corollary does not imply the uniqueness. The uniqueness holds if and only if $m = k$.

3 Exact Controllability of Stochastic Control Systems

In this section we first consider a nonlinear stochastic control system

$$dx_t = b(x_t, \sigma_t) \, dt + \sigma(x_t, \sigma_t) \, dW_t, \quad 0 \leq t \leq T. \quad (18)$$

Here $b$ and $\sigma$ are $\mathbb{R}^n$-valued functions defined on $(x, \sigma) \in \mathbb{R}^{n+1}$. We assume that $b$ and $\sigma$ are continuous with respect to $(x, \sigma)$ and uniformly Lipschitzian in $x$. We also assume that $b$ and $\sigma$ satisfy linear growth conditions in $x$ and $\sigma$. A process $u(\cdot) \in L^2_T(0, T, \mathbb{R}^n)$ is said to be an admissible control if it takes values in a given subset (constraint) $U \subset \mathbb{R}^k$. We denote by $\mathcal{U}$ the set of all admissible controls. A process $x(\cdot) \in L^2_T(0, T, \mathbb{R}^n)$ is said to be a trajectory corresponding to an admissible control $u(\cdot) \in \mathcal{U}$ if it is a solution of (18).

We now introduce some notions of controllability that measure the capacity of a stochastic control system steering a trajectory from a given initial point

$$x_0 = x \in \mathbb{R}^n \quad (19)$$

to a given terminal point

$$x_T = \xi \in L^1(\Omega, x_T, \mathbb{P}, \mathcal{M}^n). \quad (20)$$
Definition 3.1. A stochastic control system (18) is called exactly terminal-controllable if, for any \( \xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n) \), there exists at least one admissible control \( v \in \mathcal{Z} \), such that the corresponding trajectory \( x_t \) satisfies the terminal condition (20).

Definition 3.2. A stochastic control system (18) is called exactly controllable if for any \( x \in \mathbb{R}^n \) and \( \xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n) \) there exists at least one admissible control \( v \in \mathcal{Z} \), such that the corresponding trajectory \( x_t \) satisfies the initial condition (19) as well as the terminal condition (20).

In the following we only discuss the case where the control constraint is the whole space. But the conclusions may be obviously extended to more general cases.

We observe that if we regard a trajectory and the related admissible control \( (x(\cdot), v(\cdot)) \) as a pair of processes \( (y(\cdot), z(\cdot)) \) in (13), then the definition of exactly terminal controllability is just equivalent to the E-well-posedness of BSDE (13). Thus, from Proposition 2.1 and Corollary 2.1, we have immediately

Proposition 3.1. A necessary condition for a stochastic control system (18) to be exactly terminal-controllable is

\[
\begin{align*}
\{ & \text{for any } a \in \mathbb{R}^k \text{ and } p \in \mathbb{R}^n \text{ with } |p| = 1, \text{ there exists at least one pair} \\
& (x, v) \in \mathbb{R}^n \times \mathbb{R}^k, \text{ such that the scalar product } p \cdot (\sigma(x, v) - \sigma(x, a, t)) \neq 0. 
\}
\end{align*}
\]

(21)

Specially, if \( b(x, v) \) and \( \sigma(x, v) \) are Lipschitz functions of \( (x, v) \) and if \( \sigma \) is linear in \( v \),

\[
\sigma(x, v) = \sigma(x) + G_1 v,
\]

then (21) becomes

\[
\text{Rank } G_1 = n.
\]

(22) is a further necessary and sufficient condition for (18) to be exactly terminal-controllable.

We now consider the linear cases in which we can characterize the exact controllability of a system using an algebraic criterion. The control system now becomes

\[
dx_t = (F x_t + G v_t) dt + (F_1 x_t + G_1 v_t) dW_t.
\]

(23)

Here \( F \) and \( F_1 \) (resp. \( G \) and \( G_1 \)) are \( n \times n \) (resp. \( n \times k \)) matrices.

The following is a "controllability" version of Corollary 2.1.

Theorem 3.1. System (23) is exactly terminal-controllable if and only if

\[
\text{Rank } G_1 = n.
\]

In this case, we can use a simple linear transformation

\[
v = M \begin{pmatrix} z \\ u \end{pmatrix} + K x
\]
to transform (23) into an equivalent form

$$-dx_t = (Ax_t + A_1 z_t + Bu_t) dt - z_t dW_t.$$  \hspace{1cm} (24)

Here $M$ is the invertible $k \times k$-matrix introduced in the proof of Corollary 2.1, $z$ and $u$ are respectively $\mathbb{R}^n$-valued and $\mathbb{R}^{k \times n}$-valued processes.

Once (23) is written in the form of (24), we can give an algebraic criterion

**Theorem 3.2.** System (24) is exactly controllable if and only if

$$\text{Rank}[B \ AB \ A_1B \ AA_1B \ A_1AB \ \cdots] = n.$$

Before proving this theorem, we first consider the following BSDE:

$$\begin{cases} -dx_t = (Ax_t + A_1 z_t + Bu_t) dt - z_t dW_t, \\ x_T = 0. \end{cases}$$

According to Theorem 2.1, for each $u(\cdot) \in L^2_\mathcal{F}(0, T; \mathbb{R}^{k \times n})$, there exists a unique pair $(x(\cdot), z(\cdot)) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{k \times n})$ satisfying the above BSDE. We denote this pair by $(x^u(\cdot), z^u(\cdot))$. Then we have

**Lemma 3.1.** The following relation holds

$$\{x^u_0; u(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{k \times n})\} = \text{Span} \{ B \ AB \ A_1B \ AA_1B \ A_1AB \ \cdots \}.$$  \hspace{1cm} (27)

**Proof.** It is seen that $(x^u, z^u)$ depend linearly on $u$. Thus

$$\text{Span} \{x^u_0; u(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{k \times n})\} = \{x^u_0; u(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{k \times n})\}.$$

Suppose that there exists a non-zero $\beta \in \mathbb{R}^k$ such that

$$\beta \cdot x^u_0 = 0, \hspace{0.5cm} \forall u \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{k \times n}).$$

We introduce a classical Itô's linear equation starting at $\beta$

$$\begin{cases} dy_t = A^T y_t dt + \xi^T y_t dW_t, \\ y_0 = \beta. \end{cases}$$

Using Itô's formula applied to $y_t \cdot x^u_t$ yields then

$$E \int_0^T y_t \cdot Bu_t dt = x^u_0 \cdot \beta = 0,$$

or equivalently

$$E \int_u^T B^T y_t \cdot u_t dt = 0, \hspace{0.5cm} \forall u \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{k \times n}).$$

Thus

$$B^T y_t \equiv 0,$$

or

$$B^T y_t = 0,$$

(30)
\[ B^T y_i = B^T \beta + \int_0^T B^T A^T y_i \, ds + \int_0^T B^T A^T y_i \, dW_i = 0. \]

This is equivalent to
\[ B^T \beta = 0, \quad B^T A^T y_i = 0, \quad B^T A^T z_i = 0. \]

We can repeat this procedure to obtain an equivalent relation of (27)
\[ \text{Span}[B AB A_1 B AA_1 B A_1 AB, \ldots] \bot \beta \Leftrightarrow \{x^0_\infty; u(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{k-n})\} \bot \beta. \]

The proof is completed.

We now proceed to prove Theorem 3.5.

**Proof of Theorem 3.2.** According to Theorem 2.1, there exists a unique pair \((x', z')\) which solves the BSDE
\[
\begin{align*}
- dx'_i &= (Ax'_i + A_1 z'_i) \, dt - z'_i \, dW_i, \\
x'_0 &= \xi.
\end{align*}
\]

From this and BSDE (26) it follows immediately that the exact controllability of (24) is equivalent to
\[ \{x^0_\infty; u(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{k-n})\} = \mathbb{R}^n. \quad (31) \]

But, according to Lemma 3.1, Eq. (31) is equivalent to the rank condition (25). The proof is completed.

**Remark 3.1.** It is seen from Lemma 3.1 that in general case
\[ J := \text{Span}[B AB A_1 B AA_1 B A_1 AB, \ldots] \]

is the "controllable subspace" in \( \mathbb{R}^n \) for system (26). Indeed, for any \( a \in J \), there exist \((z, u) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^{k-n})\), such that the corresponding trajectory \( x \), of system (24) satisfies \( x_0 = a \).

References