A general downcrossing inequality for $g$-martingales

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Abstract

In this paper, we get a general downcrossing inequality for $g$-martingales introduced via a class of backwards stochastic differential equations (shortly BSDEs). © 2000 Elsevier Science B.V. All rights reserved

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1. Introduction

Peng (1997) introduced the notions of $g$-expectations and conditional $g$-expectations introduced via backwards stochastic differential equations (BSDEs). He showed that under suitable square integrability assumptions on the coefficient $g$ and the terminal value $\xi$, the $g$-expectation and conditional $g$-expectation of the random variable $\xi$ preserve many of the basic properties (except linearity) of the convenient mathematical expectation and conditional expectation. Using the notion of conditional $g$-expectations, Peng (1999) further introduced the notion of $g$-martingales and showed a general nonlinear Doob–Meyer decomposition theorem for $g$-supermartingales under the assumption that optional sampling theorem for $g$-supermartingales is true. In this paper, we will show a downcrossing inequality for $g$-martingales. Similar to the classical case, such a downcrossing inequality can be used to show the so-called optional sampling theorem for $g$-martingales proposed by Peng (1999).

More precisely, let $(\Omega, \mathcal{F}, P)$ be a probability space and $(W_t)_{t \geq 0}$ be a standard $d$-dimensional Brownian motion defined on this probability space. Assume that $\mathcal{F}_t = \sigma \{W_s; \ s \leq t\}$. We also assume $\mathcal{F} := \mathcal{F}_\infty = \sigma \{\bigcup_{t \geq 0} \mathcal{F}_t\}$.

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All processes mentioned in this paper are supposed to be $\{\mathcal{F}_t\}$-adapted. We use $| \cdot |$ to denote the norm of the Euclidean space $\mathbb{R}^n$. For any $t \in [0, \infty)$, let us denote by $L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$ the set of all $\mathbb{R}^n$-valued, $\mathcal{F}_t$-measurable random variables $\xi$ such that

$$E[|\xi|^2] < \infty$$

and

$$L^2_0(\Omega, \mathcal{F}, P; \mathbb{R}^n) := \bigcup_{0 \leq t < \infty} L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n).$$

Furthermore, for any $T \in [0, \infty)$, we denote by $L^2(0, T, \mathcal{F}_T, P; \mathbb{R}^n)$ the set of all $\mathbb{R}^n$-valued, $\{\mathcal{F}_t\}$-adapted processes $\{\phi_t\}$ such that

$$E\int_0^T |\phi_t|^2 \, ds < \infty$$

and

$$L^2(\mathbb{R}^n) := \bigcap_{0 \leq T < \infty} L^2(0, T, \mathcal{F}_T, P; \mathbb{R}^n).$$

Let

$$g(y, z, \cdot) : \mathbb{R} \times \mathbb{R}^d \times [0, \infty) \times \Omega \to \mathbb{R}$$

be given such that $g$ is uniformly Lipschitz in $(y, z)$, i.e., there exist two positive constants $k$ and $\mu$ such that for any $(y', z', t) \in \mathbb{R} \times \mathbb{R}^d \times [0, \infty)$ ($i = 1, 2$),

$$|g(y', z', t) - g(y^2, z^2, t)| \leq k|y^1 - y^2| + \mu|z^1 - z^2|, \quad \forall t \in [0, \infty). \tag{A.1}$$

For each $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, $g(y, z, \cdot)$ is progressively measurable. We also assume

$$g(0, 0, \cdot) \in L^2(\mathbb{R}). \tag{A.2}$$

For any given $\xi \in L^2_0(\Omega, \mathcal{F}, P; \mathbb{R})$, let $T \in [0, \infty)$ be a time horizon such that $\xi$ is $\mathcal{F}_T$-measurable. Under the assumption of (A.1) and (A.2), by the existence and uniqueness theorem (see for example El Karoui et al., 1997, Theorem 2.1), the following BSDE has a unique pair solution $(y, z) \in L^2(0, T, \mathcal{F}_T, P; \mathbb{R}) \times L^2(0, T, \mathcal{F}_T, P; \mathbb{R}^d)$ satisfying

$$y_t = \xi + \int_t^T g(y_s, z_s, s) \, ds - \int_t^T z_s \, dW_s, \quad 0 \leq t \leq T. \tag{1}$$

To shorten notations, in this paper, we will only treat the case where Brownian motion is one-dimensional: i.e. $d = 1$.

Peng (1997) introduced the following notions of $g$-expectation and conditional $g$-expectation via BSDE (1).

**Definition 1.** For each $\xi \in L^2_0(\Omega, \mathcal{F}, P; \mathbb{R})$, let $T \in [0, \infty)$ be a time horizon such that $\xi$ is $\mathcal{F}_T$-measurable. Assume that (A.1) and (A.2) hold on $g$, let $(y, z)$ be the solution of BSDE (1). We call $E^g_0, T(\xi)$ defined by

$$E^g_0, T(\xi) := y_0$$

the $g$-expectation of the random variable $\xi$ on the time interval $[0, T]$ generated by the function $g$; We call $E^g_{t, T}(\xi)$ defined by

$$E^g_{t, T}(\xi) := \begin{cases} y_t, & 0 \leq t \leq T, \\ \xi, & T < t < \infty \end{cases}$$

the conditional $g$-expectation of the random variable $\xi$ generated by the function $g$. 

The following lemma shows the relation between $g$-expectation and conditional $g$-expectation:

**Lemma 2.** If $g(0,0,\cdot) \equiv 0$, $\zeta \in L^2(Q,\mathcal{F}_T;\mathbb{R}, T \in (0,\infty)$, then there exists a unique square integrable, $\mathcal{F}_r$-measurable random variable $\eta$ such that

$$E^g_{0,r}[I_A] = E^\eta_{0,r}[\eta I_A], \quad \forall A \in \mathcal{F}_r, \quad 0 \leq r \leq T,$$

where $I_A$ is the indicator function of event $A$.

**Proof.** Since $g(0,0,\cdot) \equiv 0$, it is easy to check that, for any $A \in \mathcal{F}_r$,

$$E^g_{r,T}[I_A \zeta] = I_A E^g_{r,T}[\zeta].$$

Indeed, let $y_t := E^g_{0,T}[\zeta]$ be the solution of BSDE:

$$y_t = \zeta + \int_t^T g(y_s, z_s, s) ds - \int_t^T z_s dW_s, \quad 0 \leq t \leq T.$$

Multiplying $I_A$ on both sides of the above BSDE, note that under the assumption $g(0,0,\cdot) \equiv 0$, we have

$$E^g_{0,T}[I_A y_T] = I_A E^g_{0,T}[y_T],$$

and write $g(0,0,0) = 0$ and $g(y,z,t) = g(I_A y, I_A z, t)$ for all $(y,z,t) \in \mathbb{R} \times \mathbb{R} \times [0,\infty)$. It then follows by the fact that $A \in \mathcal{F}_r$ we have $I_A y_t \in \mathcal{F}_r$, $t \in [r,T]$, which means that $(I_A y_t, I_A z_t)$ is the unique solution of the following BSDE on the interval $[r,T]:$

$$Y_t = I_A \zeta + \int_r^T g(Y_s, Z_s, s) ds - \int_r^T Z_s dW_s, \quad r \leq t \leq T.$$

Hence $I_A y_t = Y_t$ for all $t \in [r,T]$. Since $Y_t = E^g_{r,T}[I_A \zeta]$, thus we have (E).

By the definition of $E^g_{0,T}[\cdot]$ and (E), we have

$$E^g_{0,T}[I_A \zeta] = E^\eta_{0,r}[E^g_{r,T}[I_A \zeta]] = E^\eta_{0,r}[I_A E^g_{r,T}[\zeta]], \quad 0 \leq r \leq T.$$

Set $\eta := E^g_{r,T}[I_A \zeta]$, then $\eta \in L^2(Q,\mathcal{F}_r,P)$. We now prove that $\eta$ is unique.

Indeed, suppose there exists another $\bar{\eta} \in L^2(Q,\mathcal{F}_r,P)$ such that

$$E^\eta_{0,r}[I_A \eta] = E^\bar{\eta}_{0,r}[I_A \bar{\eta}] \quad \forall A \in \mathcal{F}_r$$

(2)

but $P(\eta \neq \bar{\eta}) > 0$. Set $B := \{\eta > \bar{\eta}\}$, without loss of generality, we assume that $P(B) > 0$. It then follows by the strict Comparison Theorem (Peng, 1997, Theorem 35.3) that

$$E^\eta_{0,r}[I_B \eta] > E^\bar{\eta}_{0,r}[I_B \bar{\eta}]$$

which is in contrary with (2). \qed

For the convenience, in this paper, we write $E^{-\mu}[\zeta]$ instead of $E^g[\zeta]$ if $g(y,z,t) = -k|y| - \mu|z| - |g(0,0,t)|$ and write $\mathcal{E}^{-\mu}[\zeta]$ (resp. $\mathcal{E}^\mu[\zeta]$) instead of $E^\mu[\zeta]$ if $g(y,z,t) = -\mu|z|$ (resp. $=\mu|z|$).

Let

$$\mathcal{P} := \left\{ \mathcal{Q} : E \left[ \frac{dQ}{dP} \right]_{\mathcal{F}_T} := e^{-1/2 \int_0^T \mu^2 W^2 dt} \right\},$$

where $\mu$ is a Lipschitzian constant of the function $g$ defined in (A.1). Using the above notations, it is easy to check the following lemma:

**Lemma 3.** For any $\zeta \in L^2_0(Q,\mathcal{F},P;\mathbb{R})$, we have

$$\mathcal{E}^{-\mu}[\zeta] = \inf_{\mathcal{Q} \in \mathcal{P}} E_{\mathcal{Q}}[\zeta], \quad \mathcal{E}^\mu[\zeta] = \sup_{\mathcal{Q} \in \mathcal{P}} E_{\mathcal{Q}}[\zeta].$$
Proof. We only need to prove first equality. For any given \( \xi \in L^2(\Omega, \mathcal{F}_T, P) \), by the existence and uniqueness theorem (El Karoui et al., 1997, Theorem 2.1), let \((y_t, z_t)\) be the unique solution of BSDE:

\[
y_t = \xi - \int_t^T \mu [z_s] \, ds - \int_t^T z_s \, dW_s, \quad 0 \leq t \leq T. \tag{3}
\]

Set \( a_s := \mu \text{sgn} z_s \), then BSDE (3) can be rewritten as follows:

\[
y_t = \xi - \int_t^T z_s \, d\bar{W}_s,
\]

where \( \bar{W}_t = W_t + \int_0^t a_s \, ds \). By Girsanov’s Theorem, \( \{\bar{W}_t\} \) is \( Q^a \)-Brownian motion, where \( E[\,dQ^a/dP|\mathcal{F}_T] := e^{-1/2} \int_0^T a_s^2 \, ds - \int_0^T a_s \, dW_s \).

Taking conditional expectation \( E_Q[\cdot | \mathcal{F}_t] \) on both sides of BSDE (3)

\[
y_t = E_Q[\xi | \mathcal{F}_t] \geq \inf_{\tilde{Q} \in \mathcal{F}_t} E_Q[\xi | \mathcal{F}_t].
\]

On the other hand, let \( \{\theta_t\} \) be a process bounded by \( \mu \), i.e. \( |\theta| \leq \mu \), then the following BSDE has a unique pair solution \((y^\theta_t, z^\theta_t)\):

\[
y^\theta_t = \xi - \int_t^T \theta_s z_s ds - \int_t^T z_s dW_s, \quad 0 \leq t \leq T. \tag{4}
\]

Let \( E[\,dQ^\theta/dP|\mathcal{F}_T] := e^{-1/2} \int_0^T \theta_s^2 \, ds - \int_0^T \theta_s \, dW_s \), solving BSDE (4), we have \( y^\theta_t = E_Q[\xi | \mathcal{F}_t] \).

Note that \( \theta_s z_s \leq \mu |z_s| \), \( \forall (z, t) \in \mathbb{R}^d \times [0, T] \), comparing (3) with (4), applying Comparison Theorem (Peng, 1997, Theorem 35.3)

\[
E_Q[\xi | \mathcal{F}_t] = y^\theta_t \geq y_t,
\]

thus

\[
\text{ess inf}_{Q \in \mathcal{F}_t} E_Q[\xi | \mathcal{F}_t] \geq y_t,
\]

so \( y_t = \text{ess inf}_{Q \in \mathcal{F}_t} E_Q[\xi | \mathcal{F}_t] \). Particularly, let \( t = 0 \), note that \( y_0 = \delta^{-\mu}[\xi] \), we obtain the desired result. \( \square \)

Remark. In the definitions of \( g \)-expectation and conditional expectation, if \( g \equiv 0 \), then \( E^0_Q[\xi] = E[\xi] \); \( E^0_Q[\xi] = E[\xi|\mathcal{F}_t] \). Furthermore, Peng (1997) showed that the \( g \)-expectation and conditional \( g \)-expectation of the random variable \( \xi \) preserve many of the basic properties (except linearity) of the convenient mathematical expectation and conditional expectation.

Peng (1999) introduced the notion of weak \( g \)-martingales (Definition 4 below) and showed a Doob–Meyer decomposition theorem for strong \( g \)-martingales in the case where \( t, s \) Definition 4 are two stopping times (i.e. the optional sampling theorem is true). As we know, the downcrossing inequality for martingale is an important theorem in the proof of optional sampling theorem. In this view, the downcrossing inequality for \( g \)-martingale is useful in the proof of the so-called optional sampling theorem for \( g \)-martingales proposed by Peng (1999).

Definition 4. A real-valued adapted process \( (X_t) \) is called \( g \)-martingale (resp. \( g \)-supermartingale, \( g \)-submartingale), if \( E|X_t|^2 < \infty \quad \forall t \in [0, \infty) \) and for \( 0 \leq t \leq s < \infty \)

\[
E^g_{t, s}[X_t] = X_t \quad \text{(resp.} \leq X_t, \geq X_t).\]

Applying Comparison Theorem again, immediately.

Lemma 5. If \( \{X_t\} \) is a \( g \)-supermartingale (martingale), then \( \{X_t\} \) also is a supermartingale with respect to \( E^{-g}[\cdot] \).
2. Main results

In this section, we shall prove a downcrossing inequality for \( g \)-martingales. Let \((l_t)\) and \((u_t)\) be two processes such that \( l \leq u \); Assume \((X_t)\) is a \( g \)-supermartingale and \( 0 = t_0 < t_1 < \cdots < t_n = T \) is a strictly increasing sequence. We define \( D^\mu_{[X; n]} \) the number of downcrossings of \([l, u]\) by \( \{X_j\}_{j=0}^n \).

**Theorem 6.** Assume \((X_t)\) is a positive \( g \)-supermartingale and \( 0 = t_0, t_1, \ldots, t_n = T \) is a strictly increasing sequence. Let \( a, b \) be two positive constants such that \( a < b \). Then there exists a constant \( c > 0 \) such that the number \( D^\mu_{[X; n]} \) of downcrossings of \([a, b]\) by \( \{X_j\}_{j=0}^n \) satisfies

\[
\inf_{Q \in \mathcal{P}} E_Q[D^\mu_{[X; n]}] \leq \frac{c}{b - a} \delta^\mu[X_0 \wedge b] \tag{5}
\]

or

\[
\delta^{-\mu}[D^\mu_{[X; n]}] \leq \frac{c}{b - a} \delta^\mu[X_0 \wedge b].
\]

**Proof.** For \( j = 1, 2, \ldots, n \), let us consider the following BSDE:

\[
y^{(j)}_t = X_t - \int_t^{t_j} [k |y^{(j)}_s| + \mu|z^{(j)}_s| + |g(0, 0, s)|] \, ds - \int_t^{t_j} z^{(j)}_s \, dW_s, \quad t \in [0, t_j]. \tag{6}
\]

We define

\[
a^{(j)}_s := \begin{cases} 
-\mu \text{sgn}(z^{(j)}_s), & s \in (t_{j-1}, t_j), \\
0 & \text{otherwise},
\end{cases}
\]

\[
a := \sum_{j=1}^n a^{(j)}_s.
\]

Since \( X_t \geq 0 \), using Comparison Theorem again, we have \( y^{(j)}_t \geq 0 \), thus for each \( j = 1, 2, \ldots, n \), BSDE (6) can be rewritten by

\[
y^{(j)}_t = X_t - \int_t^{t_j} [k |y^{(j)}_s| + a \text{sgn}(z^{(j)}_s) + |g(0, 0, s)|] \, ds - \int_t^{t_j} z^{(j)}_s \, dW_s, \quad t \in [t_{j-1}, t_j]. \tag{7}
\]

Let

\[
E[dQ/dP|\mathcal{F}_T] := \exp \left\{-\frac{1}{2} \int_0^T |a_s|^2 \, ds - \int_0^T a_s \, dW(s) \right\},
\]

then \( Q \) is a probability measure on \((\Omega, \mathcal{F}_T)\).

Solving the above BSDE (7), we can obtain

\[
y^{(j)}_t = E_Q \left[ X_t e^{-k(t_{j-1}) - \int_{t_{j-1}}^t |g(0, 0, s)e^{-k(s-t)}| \, ds|\mathcal{F}_{t_{j-1}}}, \quad t \geq t_{j-1}, \tag{8}
\]

From (8) and Lemma 5,

\[
E_Q \left[ \left( X_t e^{-k(t_{j-1}) - \int_{t_{j-1}}^t |g(0, 0, s)|e^{-k(s-t)}| \, ds|\mathcal{F}_{t_{j-1}} \right) \right]
\]

\[
= y^{(j)}_t = E_{Q_{t_{j-1}, t}}^{-\mu}[X_{t_{j-1}}] \leq X_{t_{j-1}}, \quad j = 1, 2, \ldots, n.
\]

It means that the process \( Y := (Y_t)_{t=0}^T \), where \( Y_t := X_t e^{-k(t_{j-1}) - \int_{t_{j-1}}^t |g(0, 0, s)|e^{-k(s-t)} \, ds} \), is a \( Q \)-supermartingale with respect to \((\mathcal{F}_t)_{t=0}^T\).

Let \( u_t := be^{-kt} - \int_0^t |g(0, 0, s)|e^{-ks} \, ds; l_t := ae^{-kt} - \int_0^t |g(0, 0, s)|e^{-ks} \, ds \), thus \( l, u \) and \( Y \) satisfy the classical downcrossing theorem for supermartingales (see for example, Doob, 1983, p. 446):

\[
E_Q[D^\mu_{[Y; n]}] \leq e^{-kt} \frac{E_Q\{Y_0 \wedge b\}}{b - a}.
\]
It then follows by the fact that
\[ E_Q[D^n_b[Y;n]] = E_Q[D^n_b[X;n]] \]
we have
\[ E_Q[D^n_b[X;n]] \leq e^{\mu T} \frac{E_Q\{Y_0 \wedge b\}}{b-a} = e^{\mu T} \frac{E_Q\{X_0 \wedge b\}}{b-a}. \]

From this, with Lemma 3, we immediately obtain
\[ \inf_{Q \in \mathcal{F}} E_Q[D^n_b[X;n]] \leq \frac{c}{b-a} \sup_{Q \in \mathcal{F}} E_Q\{X_0 \wedge b\} \]
or
\[ \delta^{-\mu}[D^n_b[X,n]] \leq \frac{c}{b-a} \delta^\mu[X_0 \wedge b], \]

where \( c := e^{\mu T} \). The proof is complete. \( \square \)

**Corollary.** In Theorem 6, if \( g \) does not depend on \( z \), then
\[ E[D^n_b[X,n]] \leq \frac{c}{b-a} E[x \wedge b]. \]

**Proof.** If \( g \) does not depend on \( z \), we can choose \( \mu = 0 \), hence \( \delta^{-\mu}[\xi] = E[\xi] = \delta^\mu[\xi] \), from Theorem 6, we obtain the desired result. \( \square \)

As an application of the above theorems, we have the following downcrossing theorem for semi-martingales:

**Example.** Let \( \{X_t\} \) be the Itô process:
\[ X_t = x - \int_0^t g(X_s,s) \, ds + \int_0^t f(X_s,s) \, dW_s, \quad t \geq 0, \]
where \( x \) is a positive constant, \( g(y,t), f(y,t) \) are two functions defined on \( \mathbb{R} \times [0,\infty) \) such that (i) \( g, f \) satisfy Lipschitz condition in \( y \); (ii) \( g(0,\cdot) = f(0,\cdot) = 0 \).

It is easy to check that \( \{X_t\} \) is a positive \( g \)-supermartingale. Note that \( g \) does not depend on \( z \). By Corollary, we have \( E[D^n_b[X,n]] \leq c/(b-a) E[x \wedge b] \).

Using the downcrossing inequality for \( g \)-supermartingales, similar to the classical method, one can easily prove the following optional sampling theorem for \( g \)-martingales. We omit the proof.

**Theorem 7.** Let \( (X_t) \) be a right continuous positive \( g \)-supermartingale and \( 0 \leq \sigma \leq \tau \) be two bounded stopping times. Then
\[ E^{\sigma}[X_t] \leq X_\sigma \quad a.s. \]

**Remark.** If \( g \equiv 0 \), then Theorem 6 is the classical downcrossing inequality and Theorem 7 is the classical optional sampling theorem.

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