Brief Paper

Infinite horizon backward stochastic differential equation and exponential convergence index assignment of stochastic control systems

Yazeng Liu *, Shige Peng

School of Mathematics and System Sciences, Shandong University, Jinan, Shandong 250100, People’s Republic of China

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Abstract

This paper studies exponential convergence index assignment of stochastic control systems from the viewpoint of backward stochastic differential equation. Like deterministic control systems, it is shown that the exact controllability of an open-loop stochastic system is equivalent to the possibility of assigning an arbitrary exponential convergence index to the solution of the closed-loop stochastic system, formed by means of suitable linear feedback of the states. As an application, a sufficient and necessary condition for the existence and uniqueness of the solution of a class of infinite horizon forward–backward stochastic differential equations is provided. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords: Stochastic system; Stochastic differential equation; Controllability; Exponential convergence; Index assignment

1. Introduction

It is well known about pole assignment or exponential convergence index assignment of deterministic linear control systems (Wonham, 1967; Willems & Mitter, 1971). They have close relationship with the controllability of control systems. These properties are related to the very possibility of achieving the desired degree of control. A novel and interesting question is if there are similar definitions and results for stochastic control systems.

Chen (1980) studied deeply stochastic controllability and stochastic observability. Recently, study on backward stochastic differential equations (BSDE) provides a new viewpoint in this research. Peng (1994) firstly defined the exact terminal-controllability and exact controllability of stochastic control systems from the viewpoint of BSDE. He proved a necessary condition of exact terminal-controllability of nonlinear stochastic control systems. Like deterministic control systems, Peng also obtained an algebraic criterion which is a sufficient and necessary condition of exact controllability of linear stochastic control systems. Obviously, if stochastic systems degenerate to deterministic systems, the exact controllability of stochastic systems becomes the complete controllability of deterministic systems. The algebraic criterion of the exact controllability of linear stochastic control systems becomes the counterpart of the complete controllability of linear deterministic control systems.

In this paper, the following problems are discussed: (1) exponential convergence index assignment of linear stochastic control systems; (2) relationship between exponential convergence index assignment of stochastic control systems and stochastic exact controllability; and (3) relationship between exponential convergence index assignment of stochastic control systems and infinite horizon forward–backward stochastic differential equations (IHFSDE).

These problems are studied from the viewpoint of BSDE. Firstly, the existence of the solution of a class of infinite horizon backward stochastic differential equations (IHBSDE) is discussed. On this base, the exponential convergence index assignment of linear stochastic control systems is studied. Like deterministic control systems, exponential convergence index assignment of linear stochastic control systems is equivalent to exact controllability of stochastic
control systems. In addition, a sufficient and necessary condition of the existence and uniqueness of the solution of a class of IHFBSDEs is provided.

This paper is organized as follows. The main notations and preliminaries on BSDE and exact controllability of stochastic control systems are given in Section 2. Section 3 is devoted to the existence of the solution of a class of IHFBSDEs. In Section 4, exponential convergence index assignment of linear stochastic exactly terminal-controllable systems is discussed. The relationship between exponential convergence index assignment of stochastic control systems and IHFBSDEs is given in Section 5.

2. Notations and preliminaries

Let \((\Omega, \mathcal{F}, P)\) be a probability space endowed with a filtration \(\mathcal{F}_t, \ t \geq 0\) and \((W_t, \ t \geq 0)\) be a one-dimensional standard Wiener process (Brownian motion) in this space. It is assumed that the filtration \(\mathcal{F}_t\) is generated by this Wiener process \(W_t\):

\[ \mathcal{F}_t = \sigma\{W_s, \ s \leq t\}. \]

All processes mentioned in this paper are assumed to be \(\mathcal{F}_t\)-adapted. The set of all \(\mathbb{R}^m\)-valued square integrable processes, i.e.

\[ E \int_0^T |v_t|^2 \, dt < +\infty \]

is denoted by \(M^2_{\mathcal{F}}(0, T; \mathbb{R}^m)\). \(v(\cdot)\) is said in \(M^2_{\mathcal{F}}(0, T; \mathbb{R}^m)\) if \(v, \exp(ks)/2, \ s > 0\) is in \(M^2_{\mathcal{F}}(0, T; \mathbb{R}^m)\). Sometimes they will be only written as \(M^2(0, T)\) and \(M^2_k(0, T)\). Similarly, \(M^2(0, \infty)\) and \(M^2_k(0, \infty)\) are defined.

Pardoux and Peng proved the existence and uniqueness of the solution of a BSDE (Pardoux & Peng, 1990; Peng, 1991). Following this significant work, Peng (1994) pointed out that the well-posedness of BSDEs is related closely to the “exact controllability” of stochastic control systems. To be self-contained, Peng’s exact controllability of stochastic control systems is reviewed, which measures the capacity of a stochastic control system steering a trajectory from a given initial point:

\[ x_0 = x \in \mathbb{R}^n \]  

(1)
to a given terminal point:

\[ x_T = \xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n). \]  

(2)

Consider a nonlinear stochastic control system:

\[ dx_t = b(x_t, v_t) \, dt + \sigma(x_t, v_t) \, dW_t, \quad 0 \leq t \leq T, \]

(3)

where \(b\) and \(\sigma\) are \(\mathbb{R}^n\)-valued functions defined on \((x, v) \in \mathbb{R}^n \times \mathbb{R}^r\). Assume that \(b\) and \(\sigma\) are continuous with respect to \((x, v)\) and uniformly Lipschitzian in \(x\). Also assume that \(b\) and \(\sigma\) satisfy linear growth conditions in \(x\) and \(v\).

A process \(v(\cdot) \in M^2_{\mathcal{F}}(0, T; \mathbb{R}^r)\) is said to be an admissible control. The set of all admissible controls is denoted by \(\mathcal{U}\). A process \(x(\cdot) \in M^2_{\mathcal{F}}(0, T; \mathbb{R}^n)\) is said to be a trajectory corresponding to an admissible control \(v(\cdot) \in \mathcal{U}\) if it is a solution of system (3). Following definitions and results are Peng’s work (Peng, 1994).

**Definition 1.** A stochastic control system (3) is called exactly terminal-controllable if, for any \(\xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n)\), there exists at least one admissible control \(v(\cdot) \in \mathcal{U}\), such that the corresponding trajectory \(x(\cdot)\) satisfies the terminal condition (2).

**Definition 2.** A stochastic control system (3) is called exactly controllable if, for any \(x \in \mathbb{R}^n\) and \(\xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n)\), there exists at least one admissible control \(v(\cdot) \in \mathcal{U}\), such that the corresponding trajectory \(x(\cdot)\) satisfies the initial condition (1) as well as the terminal condition (2).

Consider a linear stochastic control system:

\[ dx_t = (F x_t + G v_t) \, dt + (F_1 x_t + G_1 v_t) \, dW_t, \]  

(4)

where \(F\) and \(F_1\) (resp. \(G\) and \(G_1\)) are \(n \times n\) (resp. \(n \times r\)) matrices.

**Theorem 3.** System (4) is exactly terminal-controllable if and only if

\[ \text{Rank } G_1 = n. \]

In this case, a simple linear transformation:

\[ v_t = M \begin{pmatrix} z_t - F_1 x_t \\ u_t \end{pmatrix} \]

transforms system (4) into an equivalent form:

\[ dx_t = (A x_t + A_1 z_t + B u_t) \, dt + z_t \, dW_t, \]  

(5)

where \(M\) is the invertible \(r \times r\) matrix introduced by \(G_1 M = [I_r, 0]\), \(z\) and \(u\) are respectively \(\mathbb{R}^r\)-valued and \(\mathbb{R}^{n-r}\)-valued control processes.

**Remark 4.** The control is separated to two terms, which determine the two endpoints of the trajectory, respectively. The term \(z\) controls the terminal points of the system. The term \(u\) controls the initial points of the system. From the viewpoint of BSDE, the outcomes are natural and reasonable. The term \(z\) is determined by a BSDE uniquely (Pardoux & Peng, 1990; Peng, 1991).

**Theorem 5.** System (5) is exactly controllable if and only if:

\[ \text{Rank} [B, A B, A_1 B, A A_1 B, A_1 A B, A^2 B, A_1^2 B, \ldots] = \text{Rank} [A, A_1 | B] = n. \]

Considering system (5), if control \(u\) and terminal condition \(\xi\) are deterministic, from the existence and uniqueness
Remark 6. For details of the above results see Peng’s paper (Peng, 1994).

For details of the above results see Peng’s paper (Peng, 1994).

Remark 6. According to Theorems 3 and 5, the difference between the condition to guarantee linear deterministic control systems to be completely controllable and the counterpart to guarantee linear stochastic systems to be exactly controllable is shown. The exact controllability of stochastic control systems is the extension of complete controllability of deterministic control systems with random disturbance.

3. Infinite horizon linear backward stochastic differential equation

In this section, the BSDE on an infinite time horizon is discussed, i.e. \( t \in (0, \infty) \):
\[
y_{t} = \int_{t}^{\infty} f(y_{s}, z_{s}, s) \, ds - \int_{t}^{\infty} z_{s} \, dW_{s},
\]
where \( f(y, z, s) \in M^{2}(0, \infty) \times M^{2}(0, \infty) \).

Peng (1991) proved the existence and uniqueness of the solution of the above equation.

The following infinite horizon linear BSDE is discussed in this paper:
\[
y_{t} = \int_{t}^{\infty} (A_{s}y_{s} + A_{s}z_{s} + Bu_{s}) \, ds - \int_{t}^{\infty} z_{s} \, dW_{s},
\]
where \( (y, z) \in \mathbb{R}^{n} \times \mathbb{R}^{n} \), \( u \in \mathbb{R}^{r} \), \( A, A_{s}, B \) are \( \mathbb{R}^{n \times n}, \mathbb{R}^{n \times n}, \mathbb{R}^{n \times r} \) matrices, respectively.

Theorem 7. For any given \( y \in \mathbb{R}^{n} \) and \( k \in \mathbb{R} \), there exists at least one pair \( u(\cdot) \in M^{2,k}(0, \infty; \mathbb{R}^{r}) \) such that Eq. (7) has a solution
\[
(y^{u}(\cdot), z^{u}(\cdot)) \in M^{2,k}(0, \infty) \times M^{2,k}(0, \infty), \quad y_{0}^{u} = y
\]
if and only if
\[
\text{Rank}[B, AB, A_{1}B, AA_{1}B, A_{1}AB, A_{2}B, A_{2}^{2}B, \ldots] = \text{Rank}[A, A_{1} | B] = n.
\]

Proof (Sufficiency). If \( \text{Rank}[A, A_{1} | B] = n \), according to the definition of the stochastic exact controllability, for any given \( y \in \mathbb{R}^{n} \) and \( k \in \mathbb{R} \), there exist \( T \) and control:
\[
\{ \tilde{u}_{t}, 0 \leq t \leq T \} \in M^{2,k}(0, T; \mathbb{R}^{r})
\]
such that there exists a unique pair \((y^{u}, z^{u})\) \( \in M^{2,k}(0, T) \times M^{2,k}(0, T) \) which solves the following BSDE:
\[
-dy_{t} = (A_{t}y_{t} + A_{t}z_{t} + Bu_{t}) \, dt - z_{t} \, dW_{t},
\]
\[ y_{T} = 0 \]
and \( y_{0}^{u} = y \). Let
\[
\bar{u}_{t} = \begin{cases} 
\tilde{u}_{t}, & 0 \leq t \leq T, \\
0, & t > T,
\end{cases}
\]
\[
\bar{z}_{t} = \begin{cases} 
\tilde{z}_{t}, & 0 \leq t \leq T, \\
0, & t > T,
\end{cases}
\]
for any \( k \in \mathbb{R} \), it is obvious that \((\bar{y}(\cdot), \bar{z}(\cdot)) \in M^{2,k}(0, \infty) \times M^{2,k}(0, \infty) \) and
\[
\bar{y}_{T} = \int_{T}^{\infty} (A_{s}\bar{y}_{s} + A_{s}\bar{z}_{s} + B\bar{u}_{s}) \, ds - \int_{T}^{\infty} \bar{z}_{s} \, dW_{s}, \quad \bar{y}_{0} = y.
\]

Proof (Necessity). Suppose that on the contrary, for any given \( y \in \mathbb{R}^{n} \) and \( k \in \mathbb{R} \), there exists a control:
\[
u(\cdot) \in M^{2,k}(0, \infty; \mathbb{R}^{r})
\]
such that there exists a pair \((y^{v}(\cdot), z^{v}(\cdot)) \in M^{2,k}(0, \infty) \times M^{2,k}(0, \infty) \) which solves Eq. (7) and \( y_{0}^{v} = y \), but \( \text{Rank}[A, A_{1} | B] \neq n \), i.e. there exists \( \beta \in \mathbb{R}^{n} \) and \( \beta \neq 0 \) such that
\[
\text{Span}[A, A_{1} | B] \perp \beta,
\]
where
\[
\text{Span}[A, A_{1} | B] = \text{Span}[B, AB, A_{1}B, AA_{1}B, A_{1}AB, A_{2}B, A_{2}^{2}B, \ldots].
\]

Consider the following classical Itô’s linear equation starting at \( \beta \):
\[
dx_{t} = A_{t}x_{t} \, dt + A_{t}^{\top}x_{t} \, dW_{t},
\]
\[ x_{0} = \beta.\]

According to \((y^{v}(\cdot), z^{v}(\cdot)) \in M^{2,k}(0, \infty) \times M^{2,k}(0, \infty) \), using Itô’s formula applied to \( x, y^{v} \), for any \( T > 0 \),
\[
E \int_{0}^{T} (x_{t}Bu_{t}) \, ds = \beta y_{0} - E(x_{T}y_{T}^{v}),
\]
i.e.
\[
E \int_{0}^{T} (B^{\top}x_{t}u_{t}) \, ds = \beta y_{0} - E(x_{T}y_{T}^{v}).
\]

By the Itô formula applied to \( x_{t}y_{t}^{v} \), setting \( X(t) = x_{t}y_{t}^{v} \), one obtains (following proof is from an anonymous referee):
\[
dx(t) = A_{t}x(t) + X(t)A_{t}^{\top} + A_{1}X(t)A_{1}^{\top},
\]
\[ X(0) = \beta \beta^{\top}.
\]
Let $Y$ be the subspace of $n \times n$ matrices such that $C^T Y D = 0$ for every $C, D \in \text{Span}[A, A_1 | B]$. It is noted that $Y$ is preserved by the following operations: $Y \mapsto AY$, $Y \mapsto YA^T$, $Y \mapsto A_1 YA^T$. Since $X(0) = \beta \beta^T \in Y$ by the assumption that $\beta \perp \text{Span}[A, A_1 | B]$, it follows from Eq. (9) that $X(t) \in Y$ for all $t \geq 0$. In particular, $B^T X(t) B = E(B^T x t x^T B) = 0$.

Taking the trace, it is concluded that $B^T x_t \equiv 0$. Then for any $u(\cdot) \in M^{2,1}(0, \infty; R')$ and $T > 0$
\[
E \int_0^T (x_t B u_t) \, ds = E \int_0^T (B^T x_t u_t) \, ds = 0,
\]
Choose $k \in R$ such that $E e^{-kT} |x_T|^2 \rightarrow 0$, as $T \rightarrow \infty$; then $|y_0| = |E(x_T y_T)| = |E[u(kT) y_T e^{-kT} y_T]| \leq (E e^{-kT} |x_T|^2 e^{kT} |y_T|^2)^{1/2} \rightarrow 0$, as $T \rightarrow \infty$. Therefore, for any $\beta \in R^n$, $\beta \perp y_0$. This is contradictory to $\beta \equiv 0$, then
\[
\text{Rank}[A, A_1 | B] = n.
\]
The proof is completed.

4. Exponential convergence index assignment of linear stochastic control systems

The exponential convergence index assignment of a linear exactly terminal-controllable stochastic control system is studied:
\[
dx_t = (Fx_t + Gv_t) \, dt + (F_t x_t + G_1 v_t) \, dW_t,
\]
where $x \in R^n$, $v \in R^r$, $F(F_1, G(G_1)$ are $R^{n \times n}$, $R^{n \times r}$ matrices, respectively, and $\text{Rank} G_1 = n$. It is obvious that $n$ is not larger than $r$. Then system (10) is equivalent to the following system from Theorem 3:
\[
dx_t = (Ax_t + A_1 Z_t + B u_t) \, dt + Z_t \, dW_t,
\]
where $x \in R^n$, $z \in R^n$, $u \in R^{-n}$, $A, A_1, B$ are $R^{n \times n}$, $R^{n \times n}$, $R^{n \times (r - n)}$ matrices, respectively; $(z^T, u^T)^T$ is control.

Definition 8. System (11) is called exponential convergence index assignible via closed loop if, for any given $x \in R^n$ and $k \in R$, there exist $K_1 \in R^{n \times n}$ and $K \in R^{(r - n) \times n}$ such that $x(\cdot, x, K, K_1) \in M^{2,k}(0, \infty; R^n)$,

where $x(\cdot, x, K, K_1)$ is the solution of the following system starting at $x$:
\[
dx_t = (A + A_1 K_1 + BK)x_t \, dt + K_1 x_t \, dW_t,
\]
$x_0 = x$.

Theorem 9. System (11) is exponential convergence index assignable via closed loop if and only if $\text{Rank}[A, A_1 | B] = n$, i.e. system (11) is exactly controllable.

Before proving this theorem, the following lemmas are proved. Consider the following optimal control problem:
\[
dx_t = (Ax_t + C v_t) \, dt + (B x_t + D v_t) \, dW_t,
\]
x_0 = x
\[
J^*_t(x) = \inf_{v(\cdot) \in M^{2,1}(0, T; R')} J_T(x, v(\cdot))
\]
\[
= \inf_{v(\cdot) \in M^{2,1}(0, T; R')} \int_0^T e^{\delta t} (Q x_t + v_t^T R v_t) \, dt,
\]
where $Q$ and $R$ are constant, symmetric and, respectively, nonnegative and positive definite, the constant $k$ is nonnegative.

Definition 10. System (12) is called $x$-degree of stability for initial state $x$ via open loop if, there exists $v(\cdot) \in M^{2,1}(0, \infty; R^r)$ such that $(x, x, v(\cdot)) \in M^{2,1}(0, \infty; R^n)$.

Lemma 11. The following Riccati equation has a global nonnegative solution on $[0, T]$:
\[
nP(t) = k P(t) + P(t)A + A^T P(t) + B^T P(t) B
\]
\[
- [P(t)C + B^T P(t) D][R + D^T P(t) D]^{-1}
\]
\[
[C^T P(t) + D^T P(t) B] + Q.
\]
P(T) = 0.
\[
Moreover, there are following results:
J^*_t(x) = x^T P(0) x,
\]
u^*_t = $-[R + D^T P(t) D]^{-1} [C^T P(t) + D^T P(t) B] x_t$,
where $u^*_t$ is the optimal control law.

See Bensoussan’s paper (Bensoussan, 1981) for proof. Consider the following performance index:
\[
J^*_t(x) = \inf_{v(\cdot) \in M^{2,1}(0, \infty; R')} J(x, v(\cdot))
\]
\[
= \inf_{v(\cdot) \in M^{2,1}(0, \infty; R')} E \int_0^\infty e^{\delta t} (Q x_t + v_t^T R v_t) \, dt.
\]

Lemma 12. If for any $x \in R^n$ system (12) is $k$-degree of stability via open loop, then the following Riccati equation has a nonnegative solution:
\[
k P + PA + A^T P + B^T P B
\]
\[
- [PC + B^T PD][R + D^T PD]^{-1}
\]
\[
[C^T P + D^T PB] + Q = 0.
\]
Moreover,
\[ J^*(x) = x^T P x, \tag{18} \]
\[ u^*_t = -[R + D^T P D]^{-1}(C^T P + D^T P B)x, \tag{19} \]
where \( u^*_t \) is the optimal control law.

**Proof.** (1) Existence of \( P(t) \). Let \( P(t, T, 0) \) be the solution of Eq. (13) with boundary condition \( P(T) = 0 \). Because the matrices \( A, B, C, D, Q, R \) are constant matrices, \( P(t, T, 0) \) has the following property:
\[ P(t, T, 0) = P(0, T - t, 0). \]
Since the system is \( k \)-degree of stability for any initial state, for any \( x \in \mathbb{R}^n \) there exists \( u^*(\cdot) \) such that
\[ J(x, u^*(\cdot)) < + \infty. \]
So \( P(t, T, 0) \) exists for all \( T \) and \( 0 \leq t \leq T \). Moreover, for any \( x \in \mathbb{R}^n \), \( T \) and \( 0 \leq t \leq T \)
\[ x^T P(t, T, 0)x = x^T P(0, T - t, 0)x = J_{I-t}^*(x) \leq J_{I-t}^*(x, u^*[0, T-t](\cdot)) \leq J(x, u^*(\cdot)) < + \infty. \]
It is obvious that \( P(t, T, 0) \) is monototonically increasing in \( T \), i.e.
\[ x^T P(t, T, 0)x \leq x^T P(t, T, 0)x \text{ for any } T_1 \leq T_2. \]
The bound on \( P(t, T, 0) \) together with the monotonicity relation guarantees existence of the limit \( \lim_{T \to \infty} P(t, T, 0) = P(t) \). (2) \( P(t) \) satisfies the Riccati equation.
Denote by \( P(t, T, \tilde{A}) \) the solution of Riccati equation (13) satisfying \( P(T) = \tilde{A} \). Now a moment’s reflection shows that \( P(t, T, 0) = P(t, T_1, P(T_1, T, 0)) \). For \( t \leq T_1 \leq T \)
\[ P(t) = \lim_{T \to \infty} P(t, T, 0) = \lim_{T \to \infty} P(t, T_1, P(T_1, T, 0)). \]
For fixed time \( T_1, P(t, T_1, \tilde{A}) \), the solution of Riccati equation (13) with boundary condition \( P(T) = \tilde{A} \) depends continuously on \( \tilde{A} \), therefore,
\[ P(t) = P(t, T_1, \lim_{T \to \infty} P(T_1, T, 0)) = P(t, T_1, P(T_1)) \]
which proves that \( P(t) \) is a solution of (13) defined for all \( t \). Since \( P(t) = \lim_{T \to \infty} P(t, T, 0) = \lim_{T \to \infty} P(0, T - t, 0) = P(0) \), we have \( P(t) \equiv P(0) \triangleq P \) which satisfies the Riccati equation (17).
(3) Optimal performance index and control formula.
Firstly, it is shown that the control defined by (19) is applied (where there is no assumption that this control is optimal):
\[ J(x, u^*(\cdot)) = \lim_{T \to \infty} J_T(x, u^*_0, \tau)(\cdot) = x^T P x. \tag{20} \]
In fact, direct substitution of (19) into the performance index (16), with the final time replaced by \( T \), leads to
\[ E \int_0^T e^{kT} [x_0^T Q x_t + (u^*_t)^T R u^*_t] dt = J_T(x, u^*_0, \tau)(\cdot) \]
\[ = x^T P x - E e^{kT} x^T P x_T. \]
Since \( x^T P(t, T, 0)x = x^T P(0, T - t, 0)x = J_{I-t}^*(x) \geq 0 \), \( P(t, T, 0) \) is nonnegative definite, therefore \( P \) is also nonnegative define. We have
\[ J(x, u^*(\cdot)) \leq x^T P x, \]
\[ x^T P(0, T, 0)x = J_{I-t}^*(x) \leq J_T(x, u^*_0, \tau)(\cdot) \leq J(x, u^*(\cdot)). \]
The two inequalities for \( J(x, u^*(\cdot)) \) imply Eq. (20).
It is going to show that \( u^*(\cdot) \) is an optimal control. Assuming the contrary, there is a control \( \tilde{u}(\cdot) \), different from \( u^*(\cdot) \), such that
\[ J(x, \tilde{u}(\cdot)) < J(x, u^*(\cdot)). \tag{21} \]
By (20), we have
\[ J(x, \tilde{u}(\cdot)) = \lim_{T \to \infty} J_T(x, \tilde{u}_0, \tau)(\cdot) \]
\[ \geq \lim_{T \to \infty} J_T^*(x) = x^T P x = J(x, u^*(\cdot)). \]
This is contradictory to (21). Thus \( u^*(\cdot) \) is the optimal control.

**Corollary 13.** In Lemma 12, \( P \) is positive definite if and only if
\[ \text{Rank} [A^T, B^T | M^T] = n, \tag{22} \]
where \( Q = M^T M \).

**Proof.** In fact, if \( \text{Rank} [A^T, B^T | M^T] = n \) is true and \( P \) is not positive definite, then there exists \( \tilde{x} \neq 0 \) such that \( \tilde{x}^T P \tilde{x} = 0 \), i.e.
\[ E \int_0^\infty e^{kt} \{x_0^T Q x_t + [u^*_t(\tilde{x})]^T R u^*_t(\tilde{x}) \} dt \]
\[ = E \int_0^\infty e^{kt} \{x_0^T Q x_t + [u^*_t(\tilde{x})]^T M x_t(\tilde{x}) \}
\[ + [u^*_t(\tilde{x})]^T R u^*_t(\tilde{x}) \} dt = 0. \]
So \( M x_t(\tilde{x}) \equiv 0 \) and \( u^*_t(\tilde{x}) \equiv 0 \). From the system equation (12), applying the Itô formula to \( M x_t(\tilde{x}) \equiv 0 \) repeatedly, we know that
\[ \text{Span}[A^T, B^T | M^T] \perp \tilde{x}. \]
This is contradictory to (22).
If $P$ is positive definite, suppose that on the contrary, 
$\text{Rank}[A^T,B^T|M^T] \neq n$. Then there exists $\tilde{x} \in \mathbb{R}^n$ and $\tilde{x} \neq 0$ 
such that $\text{Span}[A^T,B^T|M^T] \perp \tilde{x}$. So $Mx_\epsilon(\tilde{x},0) \equiv 0$, i.e. 
$J(\tilde{x},0) = 0 \geq J^*(\tilde{x}) = \tilde{x}^T P \tilde{x} > 0$.

This is impossible. We have proved that $P > 0$ if and only if $\text{Rank}[A^T,B^T|M^T] = n$. The proof is completed.

Now we prove Theorem 9. The proof of necessity is similar to that of Theorem 7.

Proof (Sufficiency). For any $x \in \mathbb{R}^n$ and $k > 0$, the following optimal control problem is considered:

$$
dx_t = (Ax_t + A_1z_t + Bu_t) \, dt + z_t \, dW_t,$$

$$x_0 = x,$$

$$J^*(x) = \inf_{(u(\cdot),z(\cdot)) \in \mathcal{U}} J(x,u(\cdot),z(\cdot))$$

$$= \inf_{(u(\cdot),z(\cdot)) \in \mathcal{U}} E \int_0^\infty e^{\delta t} \{ |x|^2 + |u|^2 + |z|^2 \} \, dt,$$  \hspace{1cm} (23)

where $\mathcal{U} = M^{2,k}(0,\infty;\mathbb{R}^n)$.

Since $\text{Rank}[A,A_1|B] = n$ i.e. system (11) is exactly controllable, system (11) is $k$-degree of stability via open loop 
for any initial state. From Lemma 12 and Corollary 13, we know that there exists $P > 0$ which solves the following Riccati equation:

$$kP + PA + A^T P - P(A_1,B) [I_{n+r} + (I_n,0)^T P(I_n,0)]^{-1}$$

$$(A_1,B)^T P + I_n = 0$$

and $J^*(x) = x^T P x$,

$$\begin{pmatrix} z^*_t \\ u^*_t \end{pmatrix} = -[I_{n+r} + (I_n,0)^T P(I_n,0)]^{-1}$$

$$[(A_1,B)^T P + (I_n,0) P(A_1,B)] x_t.$$

From optimal performance index, it is known that system (11) is exponential convergence index assignable via closed loop. The proof is completed.

5. Infinite horizon forward–backward stochastic differential equation and exponential convergence index assignment of stochastic control systems

The following stochastic control system and IHFBSDE are considered:

$$dx_t = (Ax_t + Cx_t) \, dt + (Bx_t + Dv_t) \, dW_t,$$  \hspace{1cm} (24)

$$x_0 = x,$$

$$dx_t = [Ax_t - C(C^T y_t + D^T z_t)] \, dt$$

$$+ [Bx_t - D(C^T y_t + D^T z_t)] \, dW_t,$$

d$y_t = [(kI + A)^T y_t + B^T z_t + x_i] \, dt - z_t \, dW_t,$$

$$x_0 = x.$$  \hspace{1cm} (25)

Theorem 14. For any $x \in \mathbb{R}^n$ and $k > 0$, there exists $v(\cdot) \in M^{2,k}(0,\infty;\mathbb{R}^r)$ such that

$$x(\cdot,x,v(\cdot)) \in M^{2,k}(0,\infty;\mathbb{R}^n),$$

where $x(\cdot,x,v(\cdot))$ is the solution of system (24) starting at $x$, if and only if for any $x \in \mathbb{R}^n$ and $k > 0$, there exists $v(\cdot) \in M^{2,k}(0,\infty;\mathbb{R}^r)$ which solves IHFBSDE (25).

Proof (Sufficiency). Suppose for any $x \in \mathbb{R}^n$ and $k > 0$, there exists

$$(x(\cdot),y(\cdot),z(\cdot)) \in M^{2,k}(0,\infty;\mathbb{R}^r)$$

which solves IHFBSDE (25). Let $v_t = -(C^T y_t + D^T z_t)$, it is obvious that: $x_0 = x$ and

$$v(\cdot) \in M^{2,k}(0,\infty;\mathbb{R}^r); \hspace{1cm} x(\cdot,x,v(\cdot)) \in M^{2,k}(0,\infty;\mathbb{R}^n).$$

Proof (Necessity). Consider the following optimal control problem:

$$dx_t = (Ax_t + Cx_t) \, dt + (Bx_t + Dv_t) \, dW_t,$$

$$x_0 = x,$$

$$J^*(x) = \inf_{v(\cdot) \in M^{2,k}(0,\infty;\mathbb{R}^r)} J(x,v(\cdot))$$

$$= \inf_{v(\cdot) \in M^{2,k}(0,\infty;\mathbb{R}^r)} E \int_0^\infty e^{\delta t} \{ |x|^2 + |v|^2 \} \, dt.$$  \hspace{1cm} (23)

Since for any $x \in \mathbb{R}^n$ and $k > 0$, there exists $v(\cdot) \in M^{2,k}(0,\infty;\mathbb{R}^r)$ such that

$$x(\cdot,x,v(\cdot)) \in M^{2,k}(0,\infty;\mathbb{R}^n).$$

From Lemma 12, the optimal control is

$$v_t = -(I + D^T PD)^{-1} (C^T P + D^T PB) x_t; \hspace{1cm} J^*(x) = x^T P x,$$

where $P$ satisfies the following Riccati equation:

$$kP + PA + AP + B^T PB - [PC + B^T PD]$$

$$[I + D^T PD]^{-1} [C^T P + D^T PB] + I = 0.$$  \hspace{1cm} (26)

Let $M = I + D^T PD, N = C^T P + D^T PB$, consider the following equation:

$$dx_t = (Ax_t - CM^{-1} N x_t) \, dt + (Bx_t - DM^{-1} N x_t) \, dW_t,$$

$$x_0 = x.$$  \hspace{1cm} (24)

It is obvious that $x(\cdot,x,v^*(\cdot)) \in M^{2,k}(0,\infty;\mathbb{R}^n)$. Since

$$dP x_t = (PA x_t - PC M^{-1} N x_t) \, dt$$

$$+ (PB x_t - PD M^{-1} N x_t) \, dW_t,$$

from Riccati equation (26), we know that

$$-dP x_t = [(kI + A)^T P x_t + x_t + B^T (PB - PD M^{-1} N x_t)] \, dt$$

$$+ (PB - PD M^{-1} N x_t) \, dW_t,$$
Let $y_t = Px_t$, $z_t = (PB - PDM^{-1}N)x_t$, it is clear that $(y(\cdot), z(\cdot)) \in M^2_k(0, \infty; \mathbb{R}^n)$

$$C^T y_t + D^T z_t$$

$$= C^T Px_t + D^T (PB - PDM^{-1}N)x_t$$

$$= (C^T P + D^T PB)x_t - (I + D^T PD)M^{-1}Nx_t + M^{-1}N x_t$$

$$= M^{-1}Nx_t = -v_t^*.$$ 

Then $(x(\cdot), y(\cdot), z(\cdot)) \in M^2_k(0, \infty; \mathbb{R}^n)$ which solves IHFBSDE (25). The proof is completed.

As an application of Theorems 9 and 14, a sufficient and necessary condition of the existence of the solution of a class of IHFBSDE is obtained. Consider the following IHFBSDE:

$$dx_t = [Ax_t - (A_1, B)(A_1, B)^T y_t - (A_1, B)(I, 0)^T z_t] dt$$

$$- (I, 0)((A_1, B)^T y_t + (I, 0)^T z_t) dW_t$$

$$= [Ax_t - (A_1A_1^T + BB^T)y_t - A_1z_t] dt - (A_1y_t + z_t) dW_t,$$

$$-dy_t = [(kl + A^T)y_t + x_t] dt - z_t dW_t,$$

$$x_0 = x. \quad (27)$$

**Theorem 15.** For any $x \in \mathbb{R}^n$ and $k \geq 0$ there exists $(x(\cdot), y(\cdot), z(\cdot)) \in M^2_k(0, \infty; \mathbb{R}^n)$ which solves IHFBSDE (27) if and only if Rank$[A, A_1 | B] = n$.

**Remark 16.** In fact, the theorem gives a method by which we ascertain the existence of solution of a class of IHFBSDEs. IHFBSDEs are new and being studied in financial mathematics. Details can be found in the work of Peng and Shi (1999, 2000).

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**References**


Yazeng Liu received his B.Sc., M.Sc. degrees in Control Theory from Shandong University, China and Ph.D. degree in Chemical Engineering (Process Control) from the University of Queensland, Australia. This work was done while he was an Associate Professor at Shandong University. He is currently a Research Fellow at the University of Birmingham, UK. His research interests include stochastic control system, wavelet-based numerical methods and model reduction, model predictive control and their applications to chemical and biological processes.

Shige Peng was awarded Thèse de 3ème Cycle de l’Université de Paris-XI, France, Thèse de Docteur de l’Université de Provence, France, Diplôme d’Habilitation à Diriger des Recherches, Université de Provence, France. Professor of Mathematics at Shandon University, PRC and Distinguished Professor of Ministry of Education of China. Invited Professor at Tsinghua University, PRC; Fudan University, PRC; Université de Rennes I, France; Université de Provence, France; Courant Institute of Mathematical Sciences, N.Y.U. USA; Brown University, USA. His main fields of interest are stochastic calculus, mathematical finance, theory of stochastic differential games, recursive utilities under risk and uncertainty, stochastic and deterministic optimal control systems, controllability of stochastic control systems, stochastic and deterministic partial differential equations, singular perturbations method for stochastic systems. He has published more than 60 international journal papers.